Last Time: Partial Fractions

Sometimes, to integrate a **rational** function, express it as *partial* fraction then solve for constants.

ex. $\int \frac{-14x+17}{10x^2+11x-6}dx = \int \frac{-14x+17}{(2x+3)(5x-2)}dx = \int \left(\frac{A}{2x+3} + \frac{B}{5x-2}\right)dx$

$= A\int \frac{1}{2x+3}dx + B\int \frac{1}{5x-2}dx$

$= -4\int \frac{1}{2x+3}dx + 3\int \frac{1}{5x-2}dx$

$= \boxed{-4\left(\frac{1}{2}\right)\ln|2x+3| + 3\left(\frac{1}{5}\right)\ln|5x-2| + C}$

Do: factor denom.

idea: use AC method?

find A, B: multiply by LCD

$\frac{-14x+17}{(2x+3)(5x-2)} = \frac{A}{2x+3} + \frac{B}{5x-2}$

$-14x+17 = A(5x-2) + B(2x+3)$

$-14x + 17 = 5Ax - 2A + 2Bx + 3B$

$(-14 = 5A + 2B)2$

$(17 = -2A + 3B)5$

created sys of eqns to solve for A, B

$\Downarrow$

$-28 = 10A + 4B$

$85 = -10A + 15B$

add to eliminate A

$\frac{57}{19} = \frac{19B}{19} \Rightarrow B = 3$:)

plug in to solve for A: $-14 = 5A + 2(3)$

$-14 = 5A + 6$

$-20 = 5A \Rightarrow A = -4$

$\int \frac{1}{2x+3}dx$

$= \frac{1}{2}\int \frac{1}{u}du$

$u = 2x+3$

$du = 2dx$

$\Downarrow$

$\frac{1}{2}du = dx$

Recall: What happens when there are more than two factors in denominator

ex. $\int \frac{x-5}{2x^3+7x^2-4x}dx = \int \frac{x-5}{x(2x-1)(x+4)}dx$

$= \int \frac{A}{x}dx + \int \frac{B}{2x-1}dx + \int \frac{C}{x+4}dx$

$= \frac{5}{4}\int \frac{1}{x}dx - 2\int \frac{1}{2x-1}dx - \frac{1}{4}\int \frac{1}{x+4}dx$

$= \boxed{\frac{5}{4}\ln|x| - \cancel{2}\left(\frac{1}{\cancel{2}}\right)\ln|2x-1| - \frac{1}{4}\ln|x+4| + C}$

solve for A, B, C

$\frac{x-5}{x(2x-1)(x+4)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+4}$ mult. each term by LCD

$x-5 = A(2x-1)(x+4) + Bx(x+4) + Cx(2x-1)$

$x=0$: $-5 = A(-1)(4)$
$-5 = -4A \Rightarrow A = \frac{5}{4}$

$\frac{5}{4}\cdot 2 = \frac{5}{2}$

$x-5 = \frac{5}{4}(2x-1)(x+4) + Bx(x+4) + Cx(2x-1)$

$x-5 = \frac{5}{4}(2x^2+7x-4) + Bx^2 + 4Bx + 2Cx^2 - Cx$

$0x^2 + x - 5 = \frac{5}{2}x^2 + \frac{35}{4}x - 5 + Bx^2 + 4Bx + 2Cx^2 - Cx$

$0 = \frac{5}{2} + B + 2C$ (x^2 terms) $\Rightarrow -\frac{5}{2} = B + 2C$

$1 = \frac{35}{4} + 4B - C$ (x terms) $2\left(-\frac{31}{4} = 4B - C\right) \Rightarrow$

$-\frac{5}{2} = B + 2C$

$-\frac{31}{2} = 8B - 2C$

$-18 = 9B$

$B = -2$

$\frac{4}{4} - \frac{35}{4} = \frac{31}{4}$

$-\frac{5}{2} = -2 + 2C$

$-\frac{5}{2} + \frac{4}{2} = 2C$

$-\frac{1}{2} = 2C \Rightarrow C = -\frac{1}{4}$

repeated factor

ex. Expand $\frac{2x^2+9x+8}{(x+2)^2(x+1)}$ to 3 terms using partial fractions technique.

$= \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+1}$

$(x+2)^2(x+1) = (x+2)(x+2)(x+1)$

factors all hv degree = 1

Recall: $(\arctan x)' = \frac{1}{1+x^2}$ **then:** $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

then $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

ex. $\int \frac{dx}{4+x^2} = \boxed{\frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$

2^2 $a=2$

Check with differentiation:

$x^2-1 = (x+1)(x-1)$

DOTS

What happens when denominator doesn't fully factor?

ex. $\int \frac{x^2-2x-3}{(x^2+1)(x-1)}\,dx = \int\left(\frac{Ax+B}{x^2+1} + \frac{C}{x-1}\right)dx = \int \frac{3x+1}{x^2+1}\,dx - 2\int \frac{1}{x-1}\,dx$

factor has degree = 2
can't factor further on reals

degree is 1

$= 3\int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - 2\ln|x-1|$

u sub

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$= 3\cdot\frac{1}{2}\int \frac{1}{u}\,du + \arctan x - 2\ln|x-1|$

$= \boxed{\frac{3}{2}\ln|x^2+1| + \arctan x - 2\ln|x-1| + C}$

solve for A, B, C

$\frac{x^2-2x-3}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$ mult by LCD

$x^2-2x-3 = (Ax+B)(x-1) + C(x^2+1)$

$x^2 - 2x - 3 = Ax^2 - Ax + Bx - B + Cx^2 + C$

$1 = A + C$ (x^2 terms)

$-2 = -A + B$ (x terms)

$-3 = C - B$ (constants)

$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

$A = 1 - C$
$A = 1-(-2)$
$\boxed{A = 3}$

$-2 = -(1-C) + B$
$-2 = -1 + C + B$
$-1 = C + B$
$-3 = C - B$
$-4 = 2C \Rightarrow \boxed{-2 = C}$

$-3 = -2 - B$
$-1 = -B$
$\boxed{B = 1}$

ex. $\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \frac{2x^2 - x + 4}{x(x^2+4)}\,dx = \int \left(\frac{A}{x} + \frac{Bx + C}{x^2+4} \right) dx$

$= A\int \frac{1}{x}\,dx + \int \frac{Bx+C}{x^2+4}\,dx$

here's the solution for constants

$2x^2 - x - 4 = A(x^2+4) + (Bx+C)\cdot x$

$2x^2 - x - 4 = Ax^2 + 4A + Bx^2 + Cx$

$2x^2 = Ax^2 + Bx^2 \Rightarrow 2 = A + B$

$-x = Cx \Rightarrow C = -1$

$-4 = 4A \Rightarrow A = -1$

$2 = -1 + B$

$3 = B$

$= -\int \frac{1}{x}\,dx + \int \frac{3x - 1}{x^2+4}\,dx$

$= -\ln|x| + 3\int \frac{x}{x^2+4}\,dx - \int \frac{1}{4+x^2}\,dx$

usub

notice

$= -\ln|x| + \frac{3}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right)$

always positive

we solved this earlier

$+C$

$= \boxed{-\ln|x| + \frac{3}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$